The BG/NBD model has proven to be a popular and robust model for customer-base analysis in continuous-time noncontractual settings. The primary reason for its popularity (relative to, say, the Pareto/NBD model) is that it is easy to implement in Excel. Hundreds of analysts and researchers have implemented the model without any problems. However, a few have come across the problem that they get a #NUM! error when evaluating the model’s likelihood function. This typically occurs in settings where some customers have made a very large number of repeat transactions. This note documents two ways of overcoming this problem.

The Problem

- The BG/NBD likelihood function for a randomly chosen customer with purchase history \((x, t_x, T)\) is

\[
L(r, \alpha, a, b \mid x, t_x, T) = \frac{B(a, b + x)}{B(a, b)} \cdot \frac{\Gamma(r + x) \alpha^r}{\Gamma(r + x + T)^{r+x}} + \delta_{x>0} \frac{B(a + 1, b + x - 1)}{B(a, b)} \cdot \frac{\Gamma(r + x) \alpha^r}{\Gamma(r + t_x)^{r+x}}.
\tag{1}
\]

- To implement the model in Excel, we rewrite (1) as

\[
L(r, \alpha, a, b \mid x, t_x, T) = A_1 \cdot A_2 \cdot (A_3 + \delta_{x>0} A_4),
\tag{2}
\]
where

\[ A_1 = \frac{\Gamma(r + x)\alpha^r}{\Gamma(r)} \quad A_2 = \frac{\Gamma(a + b)\Gamma(b + x)}{\Gamma(b)\Gamma(a + b + x)} \]

\[ A_3 = \left( \frac{1}{\alpha + T} \right)^{r + x} \quad A_4 = \left( \frac{a}{b + x - 1} \right) \left( \frac{1}{\alpha + t_x} \right)^{r + x} \]

- Taking logs gives us the following expression for the log-likelihood function:

\[
LL(r, \alpha, a, b \mid x, t_x, T) = B_1 + B_2 + \ln[\exp(B_3) + \delta_{x>0} \exp(B_4)] , \quad (3)
\]

where

\[
B_1 = r \ln(\alpha) + \ln[\Gamma(r + x)] - \ln[\Gamma(r)]
\]

\[
B_2 = \ln[\Gamma(a + b)] + \ln[\Gamma(b + x)] - \ln[\Gamma(b)] - \ln[\Gamma(a + b + x)]
\]

\[
B_3 = -(r + x) \ln(\alpha + T)
\]

\[
B_4 = \ln(a) - \ln(b + x - 1) - (r + x) \ln(\alpha + t_x)
\]

This is the expression coded-up in the Excel spreadsheet reported in the original BG/NBD paper.

- To illustrate the problem, let us consider a customer with \(x = 200, t_x = 38\) and \(T = 40\), where the unit of time is one week. For \(r = 0.25, \alpha = 4.00, a = 0.80, b = 2.50\), we get

\[
B_1 = 858.3163 \\
B_2 = -3.5458 \\
B_3 = -757.7840 \\
B_4 = -753.9973
\]

with \(LL = \#NUM!\). The problem is that \(\exp(B_3) = 7.92E-330\) and \(\exp(B_4) = 3.49E-328\), both of which are smaller than the smallest positive number that can be stored in Excel, \(2.23E-308\). As a result, Excel returns an answer of zero, and taking the log of zero in (3) gives us the \#NUM! result.

**Solution #1**

- The first solution to the problem to change the unit of time. When we change the unit of time from, say, week to day, our estimates of \(r, a\) and \(b\) will remain exactly the same, and the value of \(\alpha\) will be seven times its original value. Similarly, if we change the unit of time from week to quad-week, the value of \(\alpha\) will be a quarter of its original value.
• If we change the unit of time in the above example from week to quad-week, \( B_3 \) and \( B_4 \) will be less negative and we will typically overcome the log of zero problem. (Note that some users of the model in high-frequency environments feel the need to use day as the unit of time. This is not necessary and only increases the likelihood of facing the \#NUM! problem.)

• To illustrate this, changing the unit of time from week to quad-week sees the customer’s behavior coded as \( x = 200, t_x = 9.5 \) and \( T = 10.0 \).

With \( \alpha \) rescaled to 1.00, we get

\[
\begin{align*}
B_1 &= 857.9698 \\
B_2 &= -3.5458 \\
B_3 &= -480.1785 \\
B_4 &= -476.3918
\end{align*}
\]

with \( LL = 378.0546 \). Problem solved! (Don’t worry about the positive value of the log-likelihood function; we are not taking the log of a probability so it does not have to be negative.)

Solution #2

• Reflecting on (2) and (3), the basic problem is that we need to evaluate

\[
\ln \left[ \left( \frac{1}{\alpha + T} \right)^{r+x} + \delta_{x>0} \left( \frac{a}{b + x - 1} \right) \left( \frac{1}{\alpha + t_x} \right)^{r+x} \right].
\]

For large values of \( x \), both \( 1/(\alpha+T)^{r+x} \) and \( 1/(\alpha+t_x)^{r+x} \) are computed as zero. If we factor out \( 1/(\alpha + t_x)^{r+x} \) before taking the logs, we get

\[
-(r + x) \ln(\alpha + t_x) + \ln \left[ \left( \frac{\alpha + t_x}{\alpha + T} \right)^{r+x} + \delta_{x>0} \left( \frac{a}{b + x - 1} \right) \right].
\]

Since it is extremely unlikely that \( [(\alpha + t_x)/(\alpha + T)]^{r+x} \) will be computed as zero for large values of \( x \), the log of zero problem is avoided.

• Therefore, our second solution to the problem is to rewrite (1) as

\[
L(r, \alpha, a, b \mid x, t_x, T) = C_1 \cdot C_2 \cdot (C_3 + \delta_{x>0} C_4),
\]

where

\[
\begin{align*}
C_1 &= \frac{\Gamma(r + x)}{\Gamma(r)} \frac{\Gamma(a + b)\Gamma(b + x)}{\Gamma(b)\Gamma(a + b + x)} \\
C_2 &= \frac{\alpha^r}{(\alpha + t_x)^{r+x}} \\
C_3 &= \left( \frac{\alpha + t_x}{\alpha + T} \right)^{r+x} \\
C_4 &= \left( \frac{a}{b + x - 1} \right)
\end{align*}
\]
• Taking logs gives us the following expression for the log-likelihood function:

\[ LL(r, \alpha, a, b \mid x, t_x, T) = D_1 + D_2 + \ln(C_3 + \delta_{x>0} C_4), \]  

where

\[
D_1 = \ln[\Gamma(r + x)] - \ln[\Gamma(r)] + \ln[\Gamma(a + b)] + \ln[\Gamma(b + x)] \\
- \ln[\Gamma(b)] - \ln[\Gamma(a + b + x)] \\
D_2 = r \ln(\alpha) - (r + x) \ln(\alpha + t_x)
\]

• Let us reconsider the customer with \(x = 200, t_x = 38\) and \(T = 40\). For \(r = 0.25, \alpha = 4.00, a = 0.80, b = 2.50\), we get

\[
D_1 = 854.4240 \\
D_2 = -748.1218 \\
C_3 = 9.00E-05 \\
C_4 = 3.97E-03
\]

with \(LL = 100.7957\). Problem solved!

In Closing ...

• To summarize, the problem of getting the #NUM! error when evaluating the BG/NBD model likelihood function can be solved by rescaling time (i.e., using a longer unit of time) or using (4) instead of (3).

• The spreadsheet alt_bgnbd_ll.xlsx illustrates these two solutions for the standard CDNOW dataset.

• Note that the Pareto/NBD model suffers from exactly the same problem, which can also be overcome by either rescaling time or factoring out \(1/(\alpha + t_x)^{r+x}\) in the likelihood function.