Computing $P(X(t, t+s) = x)$ Under the BG/NBD Model

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1 Introduction

Fader et al. (2005) present an expression for $P(X(t) = x)$ given the assumptions of the BG/NBD model, where the random variable $X(t)$ denotes the number of transactions observed in the time interval $(0, t]$. In this note we derive an expression for $P(X(t, t+s) = x)$, where the random variable $X(t, t+s)$ denotes the number of transactions observed in the time interval $(t, t+s]$.

In Section 2 we review the assumptions underlying the BG/NBD model. In Section 3, we derive an expression for $P(X(t, t+s) = x)$ conditional on the individual's latent characteristics λ and p; this conditioning is removed in Section 4.

2 Model Assumptions

The BG/NBD model is based on the following six assumptions:

i) A customer's relationship with the firm has two phases: they are "alive" for an unobserved period of time, then "dead."

ii) While alive, the number of transactions made by a customer follows a Poisson process with transaction rate λ. This is equivalent to assuming that the time between transactions is distributed exponential with transaction rate λ

iii) Heterogeneity in λ follows a gamma distribution with pdf

$$f(\lambda \mid r, \alpha) = \frac{\alpha^r \lambda^{r-1} e^{-\lambda\alpha}}{\Gamma(r)}, \quad \lambda > 0\,. \tag{1}$$

iv) After any transaction, a customer dies with probability p. Therefore the point at which the customer dies is distributed *across transactions* according to a geometric distribution with pmf

$$\begin{aligned}&P(\text{die immediately after } j\text{th transaction})\\ &\qquad\qquad = p(1-p)^{j-1}, \quad j = 1, 2, 3, \ldots\,.\end{aligned}$$

v) Heterogeneity in p follows a beta distribution with pdf

$$f(p \mid a, b) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)}, \quad 0 \le p \le 1\,. \tag{2}$$

vi) The transaction rate λ and the death probability p vary independently across customers.

Two consequences of these assumptions are:

a) Conditional on λ and p, the probability that a customer is alive at time t (assuming they were alive at 0) is simply

$$\begin{aligned}P(\text{alive at } t \mid \lambda, p) &= \sum_{j=0}^{\infty} (1-p)^j \frac{(\lambda t)^j e^{-\lambda t}}{j!}\\ &= e^{-\lambda p t}\,.\end{aligned}$$

b) The time of the xth transaction is distributed Erlang-x with pdf

$$\frac{\lambda^x t^{x-1} e^{-\lambda t}}{(x-1)!}\,.$$

3 $P(X(t, t+s) = x)$ Conditional on λ and p

Let us first consider the case of $x = 0$. There are two ways no purchases could have occurred in the interval $(t, t+s]$: the individual was dead at t (with probability $1 - e^{-\lambda p t}$) or they were alive at t (with probability $e^{-\lambda p t}$) and made no purchases in that interval (with probability $e^{-\lambda s}$). Therefore,

$$P(X(t, t+s) = 0 \mid \lambda, p) = 1 - e^{-\lambda p t} + e^{-\lambda p t} e^{-\lambda s}\,. \tag{3}$$

For the case of $x > 0$, the customer had to be alive at t (with probability $e^{-\lambda pt}$). Now the xth purchase obviously occurred in the interval $(t, t+s]$, and the time of this purchase is distributed Erlang-x. What we don't know whether i) they died after their xth purchase (with probability $p(1-p)^{x-1}$) or ii) remained alive (with probability $(1-p)^x$) and made no additional purchases in the remaining time.

i) For the (unobserved) case where the customer dies after the xth purchase, we have

$$\begin{aligned} e^{-\lambda pt} p(1-p)^{x-1} \int_0^s \frac{\lambda^x u^{x-1} e^{-\lambda u}}{(x-1)!} du \\ &= e^{-\lambda pt} p(1-p)^{x-1} \left[1 - e^{-\lambda s} \sum_{j=0}^{x-1} \frac{(\lambda s)^j}{j!} \right]. \end{aligned} \tag{4}$$

ii) For the (unobserved) case where the customer remains alive after the xth purchase, we have

$$\begin{aligned} e^{-\lambda pt} (1-p)^x \int_0^s \frac{\lambda^x u^{x-1} e^{-\lambda u}}{(x-1)!} e^{-\lambda(s-u)} du \\ &= e^{-\lambda pt} (1-p)^x \lambda^x e^{-\lambda s} \int_0^s \frac{u^{x-1}}{(x-1)!} du \\ &= e^{-\lambda pt} (1-p)^x \frac{(\lambda s)^x e^{-\lambda s}}{x!}. \end{aligned} \tag{5}$$

Combining (3)–(5) gives us the following expression for the probability of observing x purchases in the interval $(t, t+s]$, conditional on λ and p:

$$\begin{aligned} P(X(t, t+s) = x \mid \lambda, p) = e^{-\lambda pt} (1-p)^x \frac{(\lambda s)^x e^{-\lambda s}}{x!} \\ &+ \delta_{x=0} \left[1 - e^{-\lambda pt} \right] \\ &+ \delta_{x>0}\, e^{-\lambda pt} p(1-p)^{x-1} \\ &- \delta_{x>0}\, e^{-\lambda pt} p(1-p)^{x-1} e^{-\lambda s} \sum_{j=0}^{x-1} \frac{(\lambda s)^j}{j!}. \end{aligned} \tag{6}$$

We note that this collapses to the basic individual-level BG/NBD pmf (Fader et al. 2005, equation 4) when $t = 0$.

4 Removing the Conditioning on λ and μ

In reality, we never know an individual's latent characteristics; we therefore remove the conditioning on λ and p by taking the expectation of (6) over the distributions of Λ and P:

$$P(X(t,t+s)=x\,|\,r,\alpha,a,b)$$
$$= \int_0^\infty \int_0^\infty P(X(t,t+s)=x\,|\,\lambda,p) g(\lambda\,|\,r,\alpha) g(p\,|\,a,b)\,d\lambda\,dp\,. \tag{7}$$

Substituting (1), (2), and (6) in (7) gives us

$$P(X(t,t+s)=x\,|\,r,\alpha,a,b) = \mathsf{A}_1 + \delta_{x=0}\,\mathsf{A}_2 + \delta_{x>0}\,\mathsf{A}_3 - \delta_{x>0}\sum_{j=0}^{x-1}\mathsf{A}_{4j}$$

where

$$\mathsf{A}_1 = \frac{s^x}{x!}\int_0^1\int_0^\infty (1-p)^x\lambda^x e^{-\lambda(s+pt)} g(\lambda\,|\,r,\alpha)\,g(p\,|\,a,b)\,d\lambda\,dp \tag{8}$$

$$\mathsf{A}_2 = 1 - \int_0^1\int_0^\infty e^{-\lambda pt} g(\lambda\,|\,r,\alpha)\,g(p\,|\,a,b)\,d\lambda\,dp \tag{9}$$

$$\mathsf{A}_3 = \int_0^1\int_0^\infty p(1-p)^{x-1} e^{-\lambda pt} g(\lambda\,|\,r,\alpha)\,g(p\,|\,a,b)\,d\lambda\,dp \tag{10}$$

$$\mathsf{A}_{4j} = \frac{s^j}{j!}\int_0^1\int_0^\infty p(1-p)^{x-1}\lambda^j e^{-\lambda(s+pt)} g(\lambda\,|\,r,\alpha)\,g(p\,|\,a,b)\,d\lambda\,dp \tag{11}$$

When solving these integrals, we will repeatedly use the result

$$\int_0^1 p^a(1-p)^b(c+dp)^{-e}\,dp$$
$$= \frac{B(a+1,b+1)}{(c+d)^e}\,{}_2F_1\big(e,b+1;a+b+2;\tfrac{d}{c+d}\big)\,. \tag{12}$$

To derive this result, let $q=1-p$ in the LHS of (12), giving us

$$\int_0^1 q^b(1-q)^a(c+d-dq)^{-e}\,dq$$
$$= (c+d)^{-e}\int_0^1 q^b(1-q)^a\big(1-\tfrac{d}{c+d}q\big)^{-e}\,dq$$

which, recalling Euler's integral representation of the Gaussian hypergeometric function (Gradshteyn and Ryzhik 2007, 9.111),

$$= \frac{B(a+1,b+1)}{(c+d)^e}\,{}_2F_1\big(e,b+1;a+b+2;\tfrac{d}{c+d}\big)\,. \tag{13}$$

Substituting (1) and (2) in (8) gives us

$$\mathsf{A}_1 = \frac{s^x}{x!}\int_0^1\left\{\int_0^\infty \frac{\alpha^r\lambda^{r+x-1}e^{-\lambda(\alpha+s+pt)}}{\Gamma(r)}\,d\lambda\right\}\frac{p^{a-1}(1-p)^{b+x-1}}{B(a,b)}\,dp$$
$$= \frac{\Gamma(r+x)}{\Gamma(r)x!}\frac{\alpha^r s^x}{B(a,b)}\int_0^1 p^{a-1}(1-p)^{b+x-1}(\alpha+s+pt)^{-(r+x)}\,dp$$

which, recalling (12),

$$= \frac{\Gamma(r+x)}{\Gamma(r)x!}\frac{B(a,b+x)}{B(a,b)}\Big(\frac{\alpha}{\alpha+t+s}\Big)^r\Big(\frac{s}{\alpha+t+s}\Big)^x \\ \times {}_2F_1\big(r+x,b+x;a+b+x;\tfrac{t}{\alpha+t+s}\big)\,.$$

Substituting (1) and (2) in (9) gives us

$$\begin{aligned} \mathsf{A}_2 &= 1-\int_0^1\bigg\{\int_0^\infty \frac{\alpha^r\lambda^{r-1}e^{-\lambda(\alpha+pt)}}{\Gamma(r)}\,d\lambda\bigg\}\frac{p^{a-1}(1-p)^{b-1}}{B(a,b)}\,dp \\ &= 1-\frac{\alpha^r}{B(a,b)}\int_0^1 p^{a-1}(1-p)^{b-1}(\alpha+pt)^{-r}\,dp \\ &= 1-\Big(\frac{\alpha}{\alpha+t}\Big)^r{}_2F_1\big(r,b;a+b;\tfrac{t}{\alpha+t}\big)\,. \end{aligned}$$

Substituting (1) and (2) in (10) gives us

$$\begin{aligned} \mathsf{A}_3 &= \int_0^1\bigg\{\int_0^\infty \frac{\alpha^r\lambda^{r-1}e^{-\lambda(\alpha+pt)}}{\Gamma(r)}\,d\lambda\bigg\}\frac{p^{a}(1-p)^{b+x-2}}{B(a,b)}\,dp \\ &= \frac{\alpha^r}{B(a,b)}\int_0^1 p^{a}(1-p)^{b+x-2}(\alpha+pt)^{-r)}\,dp \\ &= \frac{B(a+1,b+x-1)}{B(a,b)}\Big(\frac{\alpha}{\alpha+t}\Big)^r{}_2F_1\big(r,b+x-1;a+b+x;\tfrac{t}{\alpha+t}\big)\,. \end{aligned}$$

Substituting (1) and (2) in (11) gives us

$$\begin{aligned} \mathsf{A}_{4j} &= \frac{s^j}{j!}\int_0^1\bigg\{\int_0^\infty \frac{\alpha^r\lambda^{r+j-1}e^{-\lambda(\alpha+s+pt)}}{\Gamma(r)}\,d\lambda\bigg\}\frac{p^{a}(1-p)^{b+x-2}}{B(a,b)}\,dp \\ &= \frac{\Gamma(r+j)}{\Gamma(r)j!}\frac{\alpha^r s^j}{B(a,b)}\int_0^1 p^{a}(1-p)^{b+x-2}(\alpha+s+pt)^{-(r+j)}\,dp \\ &= \frac{\Gamma(r+j)}{\Gamma(r)j!}\frac{B(a+1,b+x-1)}{B(a,b)}\Big(\frac{\alpha}{\alpha+t+s}\Big)^r\Big(\frac{s}{\alpha+t+s}\Big)^j \\ &\qquad \times {}_2F_1\big(r+j,b+x-1;a+b+x;\tfrac{t}{\alpha+t+s}\big)\,. \end{aligned}$$

Bringing everything together, we have

$$P(X(t,t+s)=x\,|\,r,\alpha,a,b)=\mathsf{A}_1+\delta_{x=0}\,\mathsf{A}_2+\delta_{x>0}\,\mathsf{A}_3-\delta_{x>0}\sum_{j=0}^{x-1}\mathsf{A}_{4j} \quad (14)$$

where

$$
\begin{aligned}
\mathsf{A}_1 &= \frac{\Gamma(r+x)}{\Gamma(r)x!}\frac{B(a,b+x)}{B(a,b)}\Big(\frac{\alpha}{\alpha+t+s}\Big)^r\Big(\frac{s}{\alpha+t+s}\Big)^x \\
&\qquad \times {}_2F_1\big(r+x, b+x; a+b+x; \tfrac{t}{\alpha+t+s}\big) \\
\mathsf{A}_2 &= 1-\Big(\frac{\alpha}{\alpha+t}\Big)^r {}_2F_1\big(r, b; a+b; \tfrac{t}{\alpha+t}\big) \\
\mathsf{A}_3 &= \frac{B(a+1,b+x-1)}{B(a,b)}\Big(\frac{\alpha}{\alpha+t}\Big)^r {}_2F_1\big(r, b+x-1; a+b+x; \tfrac{t}{\alpha+t}\big) \\
\mathsf{A}_{4j} &= \frac{\Gamma(r+j)}{\Gamma(r)j!}\frac{B(a+1,b+x-1)}{B(a,b)}\Big(\frac{\alpha}{\alpha+t+s}\Big)^r\Big(\frac{s}{\alpha+t+s}\Big)^j \\
&\qquad \times {}_2F_1\big(r+j, b+x-1; a+b+x; \tfrac{t}{\alpha+t+s}\big)\,.
\end{aligned}
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Noting that ${}_2F_1\big(\cdot,\cdot;\cdot;0\big)=1$, it is clear that this collapses to the BG/NBD pmf (Fader et al. 2005, equation 8) when $t=0$.

References

Fader, Peter S., Bruce G. S. Hardie, and Ka Lok Lee (2005), “"Counting Your Customers" the Easy Way: An Alternative to the Pareto/NBD Model,” *Marketing Science*, **24** (Spring), 275–284.

Gradshteyn, I. S. and I. M. Ryzhik (2007), *Table of Integrals, Series, and Products*, Seventh Edition, Burlington, MA: Academic Press.